Int. J. Solids Structures Vol. 26, No. 3, pp. 353–368, 1990
Printed in Great Britain.
0020-7683/90 $3.00 + .00

BUCKLING EQUATIONS FOR ELASTIC SHELLS WITH ROTATIONAL DEGREES OF FREEDOM UNDERGOING FINITE STRAIN DEFORMATION†

J. MAKOWSKI
Lehrstuhl für Allgemeine Mechanik, Ruhr-Universität Bochum, 4630 Bochum, F.R.G. and Technical University, 45-223 Opole, Poland

and

H. STUMPF
Lehrstuhl für Allgemeine Mechanik, Ruhr-Universität Bochum, 4630 Bochum, F.R.G.

(*Received* 23 *January* 1989; *in revised form* 6 *June* 1989)

Abstract—A rigorous theory of small deformation superimposed on finite deformation is developed within a fully general theory of elastic shells. The mathematical structure of the configuration space and its associated tangent space is examined for the underlying shell model. Essential features of the theory are examined in the context of applications to the buckling analysis of specific problems.

1. INTRODUCTION

The buckling analysis of shells subjected to static loadings, can effectively be carried out through a linearization of the pertinent non-linear boundary value problem about given equilibrium states. The resulting linearized equations are usually referred to as buckling equations or equations of critical equilibrium and we shall adopt this terminology here. Within the classical Kirchhoff–Love type theory of shells the buckling equations have been subject of extensive investigations in the past (Koiter, 1967; Stumpf, 1981, 1984; Pietraszkiewicz *et al.*, 1984). A review of that vast literature is not our intention, nor is it relevant to the subject of this paper. It is to be noted, however, that in most previous investigations not only are strains assumed to be small but also effects due to transverse shear and transverse normal deformation of the shell are ignored. While it is expected that these effects will be negligible for thin shells made of conventional structural materials, the problem of buckling of shells undergoing finite elastic deformations cannot be adequately set down within these restrictions. It is known from the three-dimensional analysis of shell-like structures made up of highly deformable materials, like natural and synthetic rubbers or biological tissues, that buckling phenomena need not necessarily be attributed to the slenderness of the bodies but may occur in thick-walled shells as well. Moreover, many materials capable of undergoing finite elastic deformation are incompressible or nearly so. In turn, the incompressibility condition causes highly non-linear deformation through the shell thickness (cf. Appendix B). With the exception of the papers by Green and Naghdi (1971) and Zubov (1976), buckling equations for large elastic shell deformations including the aforementioned features had not been considered in the literature.

The aim of this paper is to derive the buckling equations for a fully general theory of elastic shells whose foundations have been set down by Reissner (1974), Libai and Simmonds (1983) and Simmonds (1984) *et al.* In this theory the two-dimensional equilibrium equations are obtained as exact implications of the three-dimensional balance laws of linear and angular momentum. The strain measures conjugate to the stress resultants and couples and the underlying kinematical structure of the theory are next disclosed in a natural way from the mechanical work identity being equivalent to the local equilibrium equations. In this sense the basic shell equations are exact and the resulting theory is independent of any specific physical interpretation that may be assigned to the comprised ingredients. An unavoidable approximate character, reflecting a manner in which this theory may be

† Paper presented at the XVIIth ICTAM Congress in Grenoble, France, August 1988.

expected to approximate the three-dimensional one, appears only in the constitutive relations that give the stress resultants and couples in the form of arbitrary functions of the conjugate strains. Actually, the theory of shells arrived at in this way enjoys the structure identical to that in the Cosserat shell theory. However, we have to point out that there is an ambiguity in the terminology used. Originally, E. and F. Cosserat (1909) postulated, *ab initio*, that a shell is a two-dimensional continuum to each point of which a rigid triad is attached. Later this idea has been generalized by replacing the rigid triad by three deformable directors (Ericksen and Truesdell, 1957), a single deformable director (Green *et al.*, 1965), or even an arbitrary number of deformable directors (Naghdi, 1972). The results of Reissner (1974), Libai and Simmonds (1983) and Simmonds (1984) show that the kinematical model adopted by E. and F. Cosserat (1909) is preferable to later proposed models.

In Section 2 we give a brief account of the complete set of equations for the shell modelled by the Cosserat surface. In Section 3 the underlying configuration space and its associated tangent space is constructed, concepts which are central to the subsequent developments and have not previously been investigated in the literature. In particular, we show that the configuration space for the shell modelled by the Cosserat surface is an infinite-dimensional manifold and not a linear space.

In Section 4 we derive the buckling equations as linearized equations about given finitely deformed equilibrium states of the shell. Because the configuration space is not a linear space the lack of its algebraic structure makes the process of linearization non-standard and an appeal to the advanced calculus on manifolds becomes unavoidable. In this aspect our approach parallels to that of Simo and Vu-Quoc (1986), who considered the spatial deformation of linearly elastic rods.

Finally, in Section 5 we establish sufficient conditions for the tangent operator to be symmetric, a property of major importance in the analysis of buckling problems.

2. GOVERNING EQUATIONS

We shall consider an elastostatic theory of shells whose origin is going back to E. and F. Cosserat (1909). Related and more recent contributions can be found in Reissner (1974), Libai and Simmonds (1983) and Simmonds (1984). A short summary of the governing equations presented below unifies and, in some aspects, generalizes slightly different approaches adopted in the cited papers. Our notation scheme is standard. In particular, $\mathbb{R}$ denotes the set of real numbers and $\mathscr{E}^3$ is the three-dimensional Euclidean point space whose translation space is $\mathbb{E}^3$. The elements of $\mathbb{E}^3$ are called vectors and $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ and $\mathbf{u}\otimes\mathbf{v}$ are the standard notations for the inner product, the cross product and the tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$. The elements of the vector space $L(\mathbb{E}^3, \mathbb{E}^3)$ of linear transformations of $\mathbb{E}^3$ into itself are called (second-order) tensors. The composition $\mathbf{ST}$ of two tensors $\mathbf{S}$ and $\mathbf{T}$, the transposition $\mathbf{S}^{\mathrm{T}}$ and the trace $\operatorname{tr}\mathbf{S}$ of $\mathbf{S}$ are then defined as usual. We shall also adopt the convention that lower-case Greek indices have the range 1, 2, lower-case Latin indices have the range 1, 2, 3 and that diagonally repeated indices are summed over their range. Moreover, we shall assume that various fields appearing have sufficient smoothness to justify any operations required.

By a shell we mean a thin (in some sense) three-dimensional body that may be modelled by the Cosserat surface C in its original definition (Cosserat, 1905), i.e. C comprises two ingredients, a material surface S called the carrying surface and a triad of rigid vectors called the directors. More precisely, a carrying surface is an orientable two-dimensional manifold S that can be imbedded into the physical space $\mathscr{E}^3$ by a diffeomorphism $S\to\mathscr{E}^3$. The image of S in $\mathscr{E}^3$ is a smooth surface $\bar{M}$ called the current (deformed) configuration of the carrying surface. The directors in the current configuration of the Cosserat surface C are represented by a triad $\{\mathbf{A}_i\}$ of linearly independent spatial vectors attached to each point of $\bar{M}$. We shall identify the particles of S with their material coordinates $\xi = (\xi^\alpha, \alpha = 1, 2)$. Then the current configuration of the Cosserat surface C is specified by the vector-valued functions

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}(\xi), \quad \mathbf{A}_i = \mathbf{A}_i(\xi), \quad i = 1,2,3 \tag{1}$$

where $\bar{\mathbf{r}}$ is the position vector of $\bar{M}$ relative to a fixed frame of reference.

Whenever there is no danger of confusion we may identify the carrying surface S with its initial (undeformed) configuration M whose position vector will be denoted by $\mathbf{r}(\xi)$. The directors in the initial configuration of C constitute a triad $\{\mathbf{a}_i(\xi)\}$ which we shall assume to coincide with the natural basis of $\{\xi^\alpha\}$ on M, i.e.

$$\mathbf{a}_\alpha(\xi) = \mathbf{r}_{,\alpha}, \quad \mathbf{a}_3(\xi) = \tfrac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta. \tag{2}$$

Here $\varepsilon^{\alpha\beta}$ denotes the usual permutation tensor on M, $\mathbf{a}_3$ is the unit normal vector to M and a comma indicates differentiation with respect to the material coordinates ξ^α. We next denote by $\mathbf{a}^i(\xi)$ the reciprocal base vectors, so that

$$\mathbf{a}^\alpha(\xi)\cdot\mathbf{a}_\beta(\xi) = \delta^\alpha_\beta, \quad \mathbf{a}^3(\xi) = \mathbf{a}_3(\xi), \tag{3}$$

where δ^α_β is the Kronecker symbol.

Since the directors are assumed to be rigid, the deformation of the Cosserat surface relative to the initial configuration is described by

$$\bar{\mathbf{r}}(\xi) = \mathbf{r}(\xi) + \mathbf{u}(\xi), \tag{4a}$$

$$\mathbf{A}_i(\xi) = \mathbf{Q}(\xi)\mathbf{a}_i(\xi), \quad i = 1,2,3 \tag{4b}$$

where the displacement field $\mathbf{u}$ determines the deformation of the carrying surface and $\mathbf{Q}$ is a proper orthogonal tensor (rotation tensor) specifying the deformation of the directors. In general, the deformation of the directors is independent of the deformation of the carrying surface. Also, we make no assumptions about the magnitude of the displacements, rotations and strains associated with (4). However, in order to ensure that the deformation (4) be non-singular and orientation preserving we require that

$$\det\,(\bar{\mathbf{r}}_{,\alpha}\cdot\bar{\mathbf{r}}_{,\beta}) > 0, \quad \varepsilon^{\alpha\beta}(\bar{\mathbf{r}}_{,\alpha}\times\bar{\mathbf{r}}_{,\beta})\cdot\mathbf{A}_3 > 0. \tag{5a, b}$$

The first restriction is the consequence of usual continuity assumptions while the second one implies, in particular, that $\mathbf{A}_3$ cannot be tangent to $\bar{M}$.

The suitable strain measures consist of the stretching vectors $\mathbf{E}_\beta(\xi)$ defined by

$$\mathbf{E}_\beta = \bar{\mathbf{r}}_{,\beta} - \mathbf{A}_\beta = \mathbf{u}_{,\beta} + (\mathbf{1}-\mathbf{Q})\mathbf{a}_\beta, \tag{6a}$$

and the bending vectors $\mathbf{K}_\beta(\xi)$ defined as the axial vectors of the skew-symmetric tensors $\mathscr{K}_\beta(\xi) = \mathbf{Q}_{,\beta}\mathbf{Q}^{\mathrm{T}}$, i.e.

$$\mathbf{K}_\beta \times \mathbf{v} = \mathscr{K}_\beta\mathbf{v} \quad \text{for every } \mathbf{v}\in\mathbb{E}^3. \tag{6b}$$

These strain measures have to satisfy the compatability equations (Libai and Simmonds, 1983),

$$\varepsilon^{\alpha\beta}(\mathbf{E}_{\alpha|\beta} - \mathbf{A}_\alpha\times\mathbf{K}_\beta) = \mathbf{0}, \tag{7a}$$

$$\varepsilon^{\alpha\beta}(\mathbf{K}_{\alpha|\beta} + \tfrac{1}{2}\mathbf{K}_\alpha\times\mathbf{K}_\beta) = \mathbf{0}, \tag{7b}$$

representing the integrability conditions for $\mathbf{E}_\beta$ and $\mathbf{K}_\beta$. In (7) and henceforth, $(\;)_{;\beta}$ indicates the covariant differentiation in the metric of M.

The mechanical variables entering the theory of shells modelled by the Cosserat surface C consist of the stress resultants $\mathbf{N}^\beta(\xi)$ and the couples $\mathbf{M}^\beta(\xi)$ representing the action of one part of the shell upon another along the coordinate lines ξ^β = const. on M and they are defined per unit length of these lines. The local equilibrium equations expressing the balance laws are (Libai and Simmonds, 1983).

$$\mathbf{N}^\beta|_\beta+\mathbf{p} = \mathbf{0}, \quad \mathbf{M}^\beta|_\beta+\bar{\mathbf{r}}_{,\beta}\times\mathbf{N}^\beta+\mathbf{l} = \mathbf{0}, \tag{8}$$

where $\mathbf{p}(\xi)$ and $\mathbf{l}(\xi)$ are the external surface force and couple defined per unit area of M. The equilibrium equations (8) can be obtained either by descent from three dimensions or by direct two-dimensional considerations (cf. Appendix A).

In order to formulate the associated boundary conditions we assume that M is connected with a piecewise smooth boundary ∂M whose position vector is $\mathbf{r}(s) = \mathbf{r}(\xi^\alpha(s))$. Here s denotes the arc length parameter along ∂M. At each regular point of ∂M we define the orthonormal triad $\{\mathbf{v}(s),\mathbf{t}(s),\mathbf{a}_3(s)\}$ such that

$$\mathbf{t}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = t^\alpha\mathbf{a}_\alpha, \quad t^\alpha = \frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}, \quad \mathbf{v}(s) = \mathbf{t}\times\mathbf{a}_3 = v^\alpha\mathbf{a}_\alpha, \quad v^\alpha = \varepsilon^{\alpha\beta}t_\beta \tag{9}$$

are the tangent and the outward normal vectors to ∂M lying in the tangent plane.

Now let ∂M_f denote the part of ∂M where the boundary force $\mathbf{N}_v^*(s)$ and the boundary couple $\mathbf{M}_v^*(s)$ are prescribed and let ∂M_d denote the complementary part of ∂M, i.e. $\partial M = \partial M_d \cup \partial M_f$, where the displacements $\mathbf{u}^*(s)$ and the rotations $\mathbf{Q}^*(s)$ are specified. Then the boundary conditions are

$$\mathbf{N}^\beta v_\beta = \mathbf{N}_v^*, \quad \mathbf{M}^\beta v_\beta = \mathbf{M}^* \quad \text{along } \partial M_f \tag{10a}$$

$$\mathbf{u} = \mathbf{u}^*, \qquad \mathbf{Q} = \mathbf{Q}^* \quad \text{along } \partial M_d. \tag{10b}$$

If statical and geometrical quantities are prescribed on the same part of ∂M, then they must be complementary to each other. We shall also admit that ∂M_d or even ∂M may be empty sets. In the last case the boundary conditions(10) are to be replaced by suitable periodicity conditions.

The field equations and boundary conditions given above are independent of the particular constitutive relations. Now we define the shell to be elastic if its mechanical response can be characterized by vector-valued functions $\mathcal{N}^\beta$ and $\mathcal{M}^\beta$ defined on some common domain $G\times M$ such that

$$\mathbf{N}^\beta = \mathcal{N}^\beta(\mathbf{E}_\alpha,\mathbf{K}_\alpha,\xi), \quad \mathbf{M}^\beta = \mathcal{M}^\beta(\mathbf{E}_\alpha,\mathbf{K}_\alpha;\xi). \tag{11}$$

In the hyperelastic case, the response functions $\mathcal{N}^\beta$ and $\mathcal{M}^\beta$ are given by

$$\mathcal{N}^\beta = \frac{\partial\Phi}{\partial\mathbf{E}_\beta}, \quad \mathcal{M}^\beta = \frac{\partial\Phi}{\partial\mathbf{K}_\beta}, \tag{12}$$

where the strain energy function $\Phi = \Phi(\mathbf{E}_\alpha,\mathbf{K}_\alpha;\xi)$ is defined per unit area of M. The explicit dependence of $\mathcal{N}^\beta$, $\mathcal{M}^\beta$ and Φ on ξ signifies a nonhomogeneity of the shell which may be caused by the variable curvature of M and the variable shell thickness. In the most general case the domain G of the constitutive equations is defined as a set of four-tuples $(\mathbf{E}_\alpha,\mathbf{K}_\alpha)$ at each point of which the invertibility conditions (5) are satisfied and the response functions are restricted solely by the principle of material frame-indifference and a possible material symmetry. It is to be noted, however, that if a shell modelled by the Cosserat surface C is given a specific three-dimensional interpretation the invertibility conditions will assume a more restrictive form than (5). Therefore no useful properties of G, like convexity, can be established once and for all. Moreover, the response functions must satisfy suitable restrictions to be physically reasonable to ensure the existence of solutions with the desired degree of smoothness, and yet they should admit the multiplicity of the solutions. The determination of a full set of constitutive restrictions remains as the main open problem of the shell theory [cf. Antman (1976)].

3. PRINCIPLE OF VIRTUAL DISPLACEMENTS

The Cosserat surface C manifests itself through its configurations in the physical space. A set of all configurations of C, denoted by $\mathrm{C}(C)$, is called the configuration space. In turn, each configuration represents a deformed state of the Cosserat surface and, according to (4), it is completely determined by the displacement field $\mathbf{u}$ of the carrying surface and the field of proper orthogonal tensors $\mathbf{Q}$ specifying the rigid deformation of the directors. At any fixed point $\xi \in M$ the displacement $\mathbf{u}(\xi)$ is an element of the vector space $\mathbb{E}^3$. Moreover, the set of all proper orthogonal tensors $\mathbf{Q}(\xi)$, i.e. of all isometric transformations of $\mathbb{E}^3$ into itself preserving the orientation of $\mathbb{E}^3$, constitutes the non-commutative Lie group SO(3) called the rotation group of $\mathbb{E}^3$ (Abraham and Marsden, 1978). Accordingly, the configuration space of the Cosserat surface may be defined as

$$\mathrm{C}(C) = \{\mathbb{u} = (\mathbf{u}, \mathbf{Q}) | \mathbb{u} : M \to \mathbb{E}^3 \times \mathrm{SO}(3)\}. \tag{13}$$

We require $\mathbb{u}$ to be of the class C^2 on M.

Unlike the case of particle mechanics, where the idea of a configuration space is a direct reflection of the intuitive notion of the degrees of freedom of the system, the situation we are concerned with here is more involved in a twofold sense. Firstly, the configuration space of the Cosserat surface understood as the collection of two fields $\mathbf{u}$ and $\mathbf{Q}$ is infinite dimensional. Secondly, the presence of SO(3) in the definition of $\mathrm{C}(C)$ causes the lack of the algebraic structure of linear spaces. In fact the configuration space $\mathrm{C}(C)$ can be endowed with the structure of an infinite-dimensional Riemannian manifold. However, to avoid a rather cumbersome formalism we point out that $\mathbb{u}(\xi) \in \mathbb{E}^3 \times \mathrm{SO}(3)$ at any fixed point $\xi \in M$. Clearly, $\mathbb{E}^3 \times \mathrm{SO}(3)$ enjoys the structure of the six-dimensional manifold. Moreover, it may be equipped with the group structure and thus with the structure of the Lie group. The group operation is defined by $\mathbb{u} + \mathbb{v} = (\mathbf{u}+\mathbf{v}, \mathbf{RQ})$ for any $\mathbb{u} = (\mathbf{u}, \mathbf{Q})$ and $\mathbb{v} = (\mathbf{v}, \mathbf{R})$ and the identity element is of the form $\mathbb{1} = (\mathbf{0}, \mathbf{1})$. All algebraic operations are understood here point-wise, i.e. at $\xi \in M$. An appeal to point-wise operations on the elements of the configuration space $\mathrm{C}(C)$, wherever this is feasible, will simplify our analysis considerably since the structure of $\mathbb{E}^3 \times \mathrm{SO}(3)$ is well exposed in the theory of rigid body (Abraham and Marsden, 1978).

Once the configuration space of the Cosserat surface has been defined the concept of virtual displacements may be introduced in a natural way as elements of the tangent space of $\mathrm{C}(C)$. The tangent space of a manifold generalizes the notion of tangent plane to a surface in the Euclidean space. The construction of the tangent space of $\mathrm{C}(C)$ given below is, to a large extent, analogous to its counterpart in the mechanics of rigid body, with the difference that in our case, the configuration space is infinite dimensional [cf. also Simo and Vu-Quoc (1986)].

The first notion we should introduce is that of "tangent vector" at a point $\mathbb{u} = (\mathbf{u}, \mathbf{Q})$ of $\mathrm{C}(C)$. To this end we first recall that the Lie algebra so(3) of the Lie group SO(3) is defined as the tangent space of SO(3) at the identity. Actually, so(3) is the three-dimensional vector space of all skew-symmetric tensors with the commutator taken as the Lie bracket (Marsden and Hughes, 1986), i.e. $[\mathbf{\Phi}, \mathbf{\Psi}] = \mathbf{\Phi\Psi} - \mathbf{\Psi\Phi}$ for any $\mathbf{\Phi}, \mathbf{\Psi} \in \mathrm{so}(3)$. The Lie bracket satisfies the following condition

$$[\mathbf{\Phi}, [\mathbf{\Psi}, \mathbf{\Omega}]] + [\mathbf{\Omega}, [\mathbf{\Phi}, \mathbf{\Psi}]] + [\mathbf{\Psi}, [\mathbf{\Omega}, \mathbf{\Phi}]] = \mathbf{0}, \tag{14}$$

called the Jacobi identity. The elements of so(3) are called infinitesimal generators (or infinitesimal rotations) since $\mathbf{Q}(\eta) = \exp(\eta\mathbf{\Psi})$, $\eta \in \mathbb{R}$, is a one-parameter subgroup of SO(3), i.e. $\exp(\eta\mathbf{\Psi}) \in \mathrm{SO}(3)$ for every $\mathbf{\Psi} \in \mathrm{so}(3)$. Here the exponential function $\exp : \mathrm{so}(3) \to \mathrm{SO}(3)$ is defined by

$$\exp \mathbf{\Psi} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{\Psi}^n = \mathbf{1}+\mathbf{\Psi}+\tfrac{1}{2}\mathbf{\Psi}^2+\cdots. \tag{15}$$

We also recall that another example of the Lie algebra is furnished by $\mathbb{E}^3$ with the usual cross product as the Lie bracket: $[\boldsymbol{\varphi}, \boldsymbol{\psi}] = \boldsymbol{\varphi} \times \boldsymbol{\psi}$ for any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbb{E}^3$. These two Lie algebras are isomorphic, i.e. they are indistinguishable from the mathematical point of view, with the isomorphism $\mathrm{ad} : \mathbb{E}^3 \to \mathrm{so}(3)$ called the adjoint representation of $\mathbb{E}^3$ defined by

$$(\mathrm{ad}\,\boldsymbol{\psi})\mathbf{v} = [\boldsymbol{\psi}, \mathbf{v}] \equiv \boldsymbol{\psi} \times \mathbf{v} \quad \text{for every } \mathbf{v} \in \mathbb{E}^3. \tag{16}$$

Traditionally, $\boldsymbol{\psi} \in \mathbb{E}^3$ is called the axial vector of the skew-symmetric tensor $\mathbf{\Psi} = \mathrm{ad}\,\boldsymbol{\psi} \in \mathrm{so}(3)$.

Now let $\mathbb{u} = (\mathbf{u}, \mathbf{Q}) \in \mathrm{C}(C)$ and let us consider two fields $\hat{\mathbf{u}} : M \to \mathbb{E}^3$ and $\hat{\mathbf{\Psi}} : M \to \mathrm{so}(3)$ defined on M. A mapping $\mathbb{R} \ni \eta \to \mathbb{u}(\eta) \in \mathrm{C}(C)$ defined by

$$\mathbb{u}(\eta) = (\mathbf{u}+\eta\hat{\mathbf{u}}, (\exp(\eta\hat{\mathbf{\Psi}}))\mathbf{Q}), \tag{17}$$

represents a curve on $\mathrm{C}(C)$ such that $\mathbb{u}(0) = \mathbb{u}$. The tangent vector to this curve at $\mathbb{u}$ is

$$\mathrm{D}\mathbb{u} = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbb{u}(\eta)|_{\eta=0} = (\hat{\mathbf{u}}, \hat{\mathbf{\Psi}}\mathbf{Q}) \tag{18}$$

which can easily be shown using the definition (15) of the exponential function. A set of all tangent vectors at $\mathbb{u} \in \mathrm{C}(C)$, denoted by $\mathrm{T}_{\mathbb{u}}\mathrm{C}(C)$, is called the tangent space of $\mathrm{C}(C)$ at $\mathbb{u}$. It is obvious that $\mathrm{T}_{\mathbb{u}}\mathrm{C}(C)$ is a linear space. Physically, the element $\mathrm{D}\mathbb{u} = (\hat{\mathbf{u}}, \hat{\mathbf{\Psi}}\mathbf{Q}) \in \mathrm{T}_{\mathbb{u}}\mathrm{C}(C)$ represents the infinitesimal deformation superimposed on any configuration $\mathbb{u} \in \mathrm{C}(C)$ of the Cosserat surface. Moreover, it immediately follows that the tangent space of $\mathrm{C}(C)$ at the identity $\mathbb{1} = (\mathbf{0}, \mathbf{1}) \in \mathrm{C}(C)$ is

$$\mathrm{T}_{\mathbb{1}}\mathrm{C}(C) = \{(\hat{\mathbf{u}}, \hat{\mathbf{\Psi}}) \,|\, (\hat{\mathbf{u}}, \hat{\mathbf{\Psi}}) : M \to \mathbb{E}^3 \times \mathrm{so}(3)\}. \tag{19}$$

Recalling next that the two Lie algebras so(3) and $\mathbb{E}^3$ are isomorphic, we may define the tangent space of the configuration space $\mathrm{C}(C)$ as

$$\mathrm{TC}(C) = \{\hat{\mathbb{u}} = (\hat{\mathbf{u}}, \hat{\boldsymbol{\psi}}) \,|\, \hat{\mathbb{u}} : M \to \mathbb{E}^3 \times \mathbb{E}^3\} \tag{20}$$

where $\hat{\boldsymbol{\psi}}(\xi)$ is the axial vector of the skew-symmetric tensor $\hat{\mathbf{\Psi}}(\xi)$, i.e. $\hat{\mathbf{\Psi}}(\xi) = \mathrm{ad}\,\hat{\boldsymbol{\psi}}(\xi)$.

For elements of the tangent space (20) we introduce the notion of generalized virtual displacements understood in the sense of a collection of the virtual displacement of the carrying surface and the virtual rotation of the directors.

Now let $\mathbb{u} = (\mathbf{u}, \mathbf{Q}) \in \mathrm{C}(C)$ be an arbitrary configuration of the Cosserat surface and let $\hat{\mathbb{u}} = (\hat{\mathbf{u}}, \hat{\boldsymbol{\psi}}) \in \mathrm{TC}(C)$ denote kinematical admissible generalized virtual displacements, i.e. $\hat{\mathbb{u}} = \mathbb{o}$ along ∂M_d. Consider the following functional

$$\Pi[\mathbb{u}; \hat{\mathbb{u}}] = \iint_M \{\mathbf{N}^\beta \cdot (\hat{\mathbf{u}}_{,\beta} - \hat{\boldsymbol{\psi}} \times \bar{\mathbf{r}}_{,\beta}) + \mathbf{M}^\beta \cdot \hat{\boldsymbol{\psi}}_{,\beta}\}\,\mathrm{d}A - \iint_M (\mathbf{p}\cdot\hat{\mathbf{u}} + \mathbf{l}\cdot\hat{\boldsymbol{\psi}})\,\mathrm{d}A - \int_{\partial M_f} (\mathbf{N}^*_\nu \cdot \hat{\mathbf{u}} + \mathbf{M}^*_\nu \cdot \hat{\boldsymbol{\psi}})\,\mathrm{d}s, \tag{21}$$

where the stress resultants $\mathbf{N}^\beta$ and the couples $\mathbf{M}^\beta$ are to be regarded as functions of $\mathbb{u}$ through the constitutive equations (11) and the kinematical relations (6). We also admit the external surface and boundary loadings to be configuration dependent, i.e. $\mathbf{p}$, $\mathbf{l}$, $\mathbf{N}_\nu^*$ and $\mathbf{M}_\nu^*$ may depend on $\mathbb{u}$. The functional (21) is linear in $\hat{\mathbb{u}}$ but depends non-linearly on $\mathbb{u}$.

Using Stokes' theorem the functional (21) can be transformed yielding

$$I[\mathbb{u};\hat{\mathbb{u}}] = -\iint_M \{(\mathbf{N}^\beta|_\beta+\mathbf{p})\cdot\hat{\mathbf{u}}+(\mathbf{M}^\beta|_\beta+\bar{\mathbf{r}}_{,\beta}\times\mathbf{N}^\beta+\mathbf{l})\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}A$$

$$+\int_{\partial M_f}\{(\mathbf{N}^\beta\nu_\beta-\mathbf{N}_\nu^*)\cdot\hat{\mathbf{u}}+(\mathbf{M}^\beta\nu_\beta-\mathbf{M}_\nu^*)\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}s+\int_{\partial M_d}\{(\mathbf{N}^\beta\nu_\beta)\cdot\hat{\mathbf{u}}+(\mathbf{M}^\beta\nu_\beta)\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}s. \quad (22)$$

The second line integral in (22) vanishes for $\hat{\mathbb{u}}$ must be kinematically admissible. Moreover, from (22) it follows that $\mathbb{u}\in C(C)$ is an equilibrium configuration satisfying the equilibrium equations (8) and the static boundary condition (10a) if, and only if,

$$I[\mathbb{u};\hat{\mathbb{u}}] = 0,\quad \forall\,\hat{\mathbb{u}}\in TC(C),\quad \hat{\mathbb{u}} = \mathbb{o} \text{ along } \partial M_d. \quad (23)$$

4. BUCKLING EQUATIONS

In the previous section we regarded the elements of the tangent space TC (C) as virtual displacements. Actually, the tangent space models locally a configuration space. Most linearizations in physics consist of replacing a given configuration space by its tangent space at a point. This concept will be used to derive the buckling equations, i.e. the linearized equations about a given equilibrium state of the shell.

Let $\mathbb{u} = (\mathbf{u},\mathbf{Q})\in C(C)$ be an equilibrium configuration of the Cosserat surface and let, as before, $\hat{\mathbb{u}} = (\hat{\mathbf{u}},\hat{\boldsymbol{\psi}})\in TC(C)$ denote a kinematically admissible virtual displacement. Consider now another kinematically admissible element of the tangent space which we denote by $\breve{\mathbb{u}} = (\breve{\mathbf{u}},\breve{\boldsymbol{\psi}})$, $\breve{\mathbb{u}}\in TC(C)$. We may regard $\breve{\mathbb{u}}$ as the infinitesimal generator of a one-parameter family of configurations represented by a curve,

$$\mathbb{u}(\eta) = (\mathbf{u}+\eta\breve{\mathbf{u}},\ \exp(\eta\breve{\boldsymbol{\Psi}})\mathbf{Q}),\quad \eta\geqslant 0 \quad (24)$$

on the configuration space and such that $\mathbb{u}(0) = \mathbb{u}$. Here, in accordance with our notation scheme, $\breve{\boldsymbol{\Psi}} = \mathrm{ad}\,\breve{\boldsymbol{\psi}}$. In physical terms, $\breve{\mathbb{u}}$ represents a small deformation superimposed on the equilibrium configuration $\mathbb{u}$. By virtue of the principle of virtual displacements (23) the configurations of the shell represented by (24) are equilibrium ones if and only if:

$$I[\mathbb{u}(\eta);\hat{\mathbb{u}}] = 0,\quad \forall\,\hat{\mathbb{u}}\in TC(C),\quad \hat{\mathbb{u}} = 0 \text{ along } \partial M_d. \quad (25)$$

The buckling equations can now be obtained by linearization of the functional $I[\mathbb{u}(\eta);\hat{\mathbb{u}}]$ in the principle of virtual displacements (25) about the equilibrium configuration $\mathbb{u}$, i.e.

$$DI[\mathbb{u};\breve{\mathbb{u}},\hat{\mathbb{u}}] = \frac{\mathrm{d}}{\mathrm{d}\eta}I[\mathbb{u}(\eta);\hat{\mathbb{u}}]|_{\eta=0} = 0. \quad (26)$$

Hence, it only remains to calculate the first differential of the functional (21).

According to (24), the position vector of the carrying surface and the directors in the configuration $\mathbb{u}(\eta)$ of the shell are given by

$$\bar{\mathbf{r}}(\eta) = \bar{\mathbf{r}}+\eta\breve{\mathbf{u}}, \quad (27)$$

$$\mathbf{A}_i(\eta) = (\exp(\eta\breve{\boldsymbol{\Psi}}))\mathbf{A}_i = (\mathbf{1}+\eta\breve{\boldsymbol{\Psi}}+\cdots)\mathbf{A}_i = \mathbf{A}_i+\eta\breve{\boldsymbol{\psi}}\times\mathbf{A}_i+\cdots. \quad (28)$$

The associated strains we denote by $\mathbf{E}_\beta(\eta)$ and $\mathbf{K}_\beta(\eta)$. Then the corresponding stresses $\mathbf{N}^\beta(\eta)$ and $\mathbf{M}^\beta(\eta)$ are determined by the constitutive equations (11). Adopting the same notation convention for the external loadings it follows from (21)

$$I[\mathfrak{u}(\eta);\hat{\mathfrak{u}}] = \iint_M \{\mathbf{N}^\beta(\eta)\cdot[\hat{\mathbf{u}}_{,\beta} - \hat{\boldsymbol{\psi}}\times(\bar{\mathbf{r}}_{,\beta}+\eta\check{\mathbf{u}}_{,\beta})] + \mathbf{M}^\beta(\eta)\cdot\hat{\boldsymbol{\psi}}_{,\beta} - \mathbf{p}(\eta)\cdot\hat{\mathbf{u}} - \mathbf{l}(\eta)\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}A - \int_{\partial M_f}\{\mathbf{N}^*_\nu(\eta)\cdot\hat{\mathbf{u}} + \mathbf{M}^*_\nu(\eta)\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}s, \tag{29}$$

and hence

$$\mathrm{D}I[\mathfrak{u};\check{\mathfrak{u}},\hat{\mathfrak{u}}] = \iint_M \{\mathrm{D}\mathbf{N}^\beta\cdot(\hat{\mathbf{u}}_{,\beta} - \hat{\boldsymbol{\psi}}\times\bar{\mathbf{r}}_{,\beta}) - \mathbf{N}^\beta\cdot(\hat{\boldsymbol{\psi}}\times\check{\mathbf{u}}_{,\beta}) + \mathrm{D}\mathbf{M}^\beta\cdot\hat{\boldsymbol{\psi}}_{,\beta} - \mathrm{D}\mathbf{p}\cdot\hat{\mathbf{u}} - \mathrm{D}\mathbf{l}\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}A - \int_{\partial M_f}(\mathrm{D}\mathbf{N}^*_\nu\cdot\hat{\mathbf{u}} + \mathrm{D}\mathbf{M}^*_\nu\cdot\hat{\boldsymbol{\psi}})\,\mathrm{d}s. \tag{30}$$

Here DF is the short notation for the directional derivative $\mathrm{D}\mathbf{F}[\mathfrak{u};\check{\mathfrak{u}}]$ of the quantity $\mathbf{F}$ evaluated at $\mathfrak{u}\in \mathrm{C}(C)$ in the direction $\check{\mathfrak{u}}\in \mathrm{TC}(C)$:

$$\mathrm{D}\mathbf{F}[\mathfrak{u};\check{\mathfrak{u}}] = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{F}[\mathfrak{u}(\eta)]|_{\eta=0}. \tag{31}$$

In particular, from (27) and (28) we obtain

$$\mathrm{D}\bar{\mathbf{r}} = \check{\mathbf{u}}, \quad \mathrm{D}\mathbf{A}_i = \check{\boldsymbol{\psi}}\times\mathbf{A}_i. \tag{32}$$

If $\mathbf{F}$ represents a vector field defined through its components with respect to the rotated base $\mathbf{A}_i$, as the internal stresses $\mathbf{N}^\beta$, $\mathbf{M}^\beta$ or the external loadings $\mathbf{p}$, $\mathbf{l}$, $\mathbf{N}^*_\nu$, $\mathbf{M}^*_\nu$, then $\mathbf{F}(\eta) = F^i(\eta)\mathbf{A}_i(\eta)$ and with the use of (32) one gets

$$\mathrm{D}\mathbf{F} = \mathrm{D}F^i\mathbf{A}_i + F^i\mathrm{D}\mathbf{A}_i = \mathrm{D}F^i\mathbf{A}_i + F^i\check{\boldsymbol{\psi}}\times\mathbf{A}_i, \tag{33}$$

and hence

$$\check{\mathbf{F}} \equiv \mathrm{D}F^i\mathbf{A}_i = \mathrm{D}\mathbf{F} - \check{\boldsymbol{\psi}}\times\mathbf{F} \tag{34}$$

may be called the corotational differential of the field $\mathbf{F}$. Here again $\check{\mathbf{F}}$ is short for $\check{\mathbf{F}}[\mathfrak{u};\check{\mathfrak{u}}] = \mathrm{D}F^i[\mathfrak{u};\check{\mathfrak{u}}]\mathbf{A}_i$. For later use we note that in the case of a tensor field $\mathbf{T} = T^{ij}\mathbf{A}_i\otimes\mathbf{A}_j$ the corotational differential of $\mathbf{T}$ is given by

$$\check{\mathbf{T}} \equiv \mathrm{D}T^{ij}\mathbf{A}_i\otimes\mathbf{a}_j = \mathrm{D}\mathbf{T} - \check{\boldsymbol{\Psi}}\mathbf{T} + \mathbf{T}\check{\boldsymbol{\Psi}}, \quad \check{\boldsymbol{\Psi}} = \mathrm{ad}\,\check{\boldsymbol{\psi}}. \tag{35}$$

In particular, if $\mathbf{T}$ is a field with values in so(3) then $\check{\mathbf{T}} = \mathrm{D}\mathbf{T} - [\check{\boldsymbol{\Psi}},\mathbf{T}]$, where [,] denotes the Lie bracket. The formula (35) can be shown by direct use of the definition (31), the relation (32) and the following identities $\mathbf{T}(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{T}\mathbf{u})\otimes\mathbf{v}$, $(\mathbf{u}\otimes\mathbf{v})\mathbf{T} = \mathbf{u}\otimes(\mathbf{T}^{\mathrm{T}}\mathbf{v})$ valid for any second-order tensor $\mathbf{T}$ and any vectors $\mathbf{u}$ and $\mathbf{v}$.

Now, differentiation of the constitutive relations (11) with the use of the formula (34) yields

$$\begin{aligned}\check{\mathbf{N}}^\alpha &= \mathbf{C}_1^{\alpha\beta}\check{\mathbf{E}}_\beta + \mathbf{C}_2^{\alpha\beta}\check{\mathbf{K}}_\beta,\\ \check{\mathbf{M}}^\alpha &= \mathbf{C}_3^{\alpha\beta}\check{\mathbf{E}}_\beta + \mathbf{C}_4^{\alpha\beta}\check{\mathbf{K}}_\beta,\end{aligned} \tag{36}$$

where the second-order tensors (elasticity tensors) $\mathbf{C}_p^{\alpha\beta} = \mathbf{C}_p^{\alpha\beta}(\mathbf{E}_\lambda, \mathbf{K}_\lambda; \xi)$, $p = 1,2,3,4$, are defined by

$$\mathbf{C}_1^{\alpha\beta} = \frac{\partial \mathcal{N}^\alpha}{\partial \mathbf{E}_\beta}, \quad \mathbf{C}_2^{\alpha\beta} = \frac{\partial \mathcal{N}^\alpha}{\partial \mathbf{K}_\beta},$$
$$\mathbf{C}_3^{\alpha\beta} = \frac{\partial \mathcal{M}^\alpha}{\partial \mathbf{E}_\beta}, \quad \mathbf{C}_4^{\alpha\beta} = \frac{\partial \mathcal{M}^\alpha}{\partial \mathbf{K}_\beta}. \tag{37}$$

In the case of a hyperelastic shell the elasticity tensors are given by the second partial derivatives of the strain energy function $\Phi(\mathbf{E}_\alpha, \mathbf{K}_\alpha; \xi)$ and they enjoy the following symmetry conditions

$$\mathbf{C}_p^{\alpha\beta} = (\mathbf{C}_p^{\beta\alpha})^{\mathrm{T}}, \quad p = 1,4, \quad \alpha,\beta = 1,2$$
$$\mathbf{C}_2^{\alpha\beta} = (\mathbf{C}_3^{\beta\alpha})^{\mathrm{T}}, \quad \alpha,\beta = 1,2. \tag{38}$$

The corotational differentials $\check{\mathbf{E}}_\beta$, $\check{\mathbf{K}}_\beta$ of the strain measures entering the linearized constitutive equations (36) have the form

$$\check{\mathbf{E}}_\beta = \check{\mathbf{u}}_{,\beta} - \check{\boldsymbol{\psi}} \times \bar{\mathbf{r}}_{,\beta}, \quad \check{\mathbf{K}}_\beta = \check{\boldsymbol{\psi}}_{,\beta}. \tag{39a, b}$$

To show this, we note that in view of (27), (28) and (6)

$$\mathbf{E}_\beta(\eta) = \bar{\mathbf{r}}_{,\beta}(\eta) - \mathbf{A}_\beta(\eta) = \mathbf{E}_\beta + \eta(\check{\mathbf{u}}_{,\beta} - \check{\boldsymbol{\psi}} \times \mathbf{A}_\beta) + \cdots \tag{40}$$

and hence

$$\mathrm{D}\mathbf{E}_\beta = \check{\mathbf{u}}_{,\beta} - \check{\boldsymbol{\psi}} \times \mathbf{A}_\beta = \check{\mathbf{u}}_{,\beta} - \check{\boldsymbol{\psi}} \times \bar{\mathbf{r}}_{,\beta} + \check{\boldsymbol{\psi}} \times \mathbf{E}_\beta. \tag{41}$$

From (41) and the formula (34) the relation (39a) follows. Next, setting

$$\mathbf{Q}(\eta) = (\exp(\eta\check{\boldsymbol{\Psi}}))\mathbf{Q} = (\mathbf{1} + \eta\check{\boldsymbol{\Psi}} + \cdots)\mathbf{Q} \in \mathrm{SO}(3) \tag{42}$$

at any $\xi \in M$, by the definition (6b) of the bending vectors we have $\mathscr{K}_\beta(\eta) = \mathrm{ad}\,\mathbf{K}_\beta(\eta)$, where $\mathscr{K}_\beta(\eta) \in \mathrm{so}(3)$ is given by

$$\mathscr{K}_\beta(\eta) = \mathbf{Q}_{,\beta}(\eta)\mathbf{Q}^{\mathrm{T}}(\eta) = \mathscr{K}_\beta + \eta(\check{\boldsymbol{\Psi}}_{,\beta} + [\check{\boldsymbol{\Psi}}, \mathscr{K}_\beta]) + \cdots. \tag{43}$$

Differentiation of (43) yields

$$\mathrm{d}\mathscr{K}_\beta = \check{\boldsymbol{\Psi}}_{,\beta} + [\check{\boldsymbol{\Psi}}, \mathscr{K}_\beta]. \tag{44}$$

Clearly, $\mathrm{D}\mathscr{K}_\beta \in \mathrm{so}(3)$ at any $\xi \in M$. Moreover, we have $\mathrm{D}\mathscr{K}_\beta = \mathrm{ad}\,(\mathrm{D}\mathbf{K}_\beta)$, which in view of (44) implies that

$$\mathrm{D}\mathbf{K}_\beta = \check{\boldsymbol{\psi}}_{,\beta} + \check{\boldsymbol{\psi}} \times \mathbf{K}_\beta, \tag{45}$$

and hence (39b) follows.

Having established the linearized constitutive equations and the linearized kinematical relations we are now ready to derive the remaining linearized field equations and the boundary conditions. To this end we apply the formula (34) to the differentials of the stresses and the external loadings entering (30) yielding

$$\begin{aligned}\mathrm{D}I[\mathfrak{u};\check{\mathfrak{u}},\hat{\mathfrak{u}}] = \iint_M \{&\check{\mathbf{N}}^\beta\cdot(\hat{\mathbf{u}}_{,\beta}-\hat{\boldsymbol{\psi}}\times\bar{\mathbf{r}}_{,\beta})+\check{\mathbf{M}}^\beta\cdot\hat{\boldsymbol{\psi}}_{,\beta}+\mathbf{N}^\beta\cdot(\check{\mathbf{u}}_{,\beta}\times\hat{\boldsymbol{\psi}})\\ &+(\check{\boldsymbol{\psi}}\times\mathbf{N}^\beta)\cdot(\hat{\mathbf{u}}_{,\beta}-\hat{\boldsymbol{\psi}}\times\bar{\mathbf{r}}_{,\beta})+(\check{\boldsymbol{\psi}}\times\mathbf{M}^\beta)\cdot\hat{\boldsymbol{\psi}}_{,\beta}\\ &-(\check{\mathbf{p}}+\check{\boldsymbol{\psi}}\times\mathbf{p})\cdot\hat{\mathbf{u}}-(\check{\mathbf{l}}+\check{\boldsymbol{\psi}}\times\mathbf{l})\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}A\\ &-\int_{\partial M_f}\{(\check{\mathbf{N}}_\nu^*+\check{\boldsymbol{\psi}}\times\mathbf{N}_\nu^*)\cdot\hat{\mathbf{u}}+(\check{\mathbf{M}}_\nu^*+\check{\boldsymbol{\psi}}\times\mathbf{M}_\nu^*)\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}s. \end{aligned} \tag{46}$$

It is to be noted here that $\check{\mathbf{p}}$, $\check{\mathbf{l}}$, $\check{\mathbf{N}}_\nu^*$ and $\check{\mathbf{M}}_\nu^*$ are, by their definitions, linear in $\check{\mathfrak{u}}$ at the most. Applying now the Stokes' theorem to (46) and subsequently making use of the identity

$$\bar{\mathbf{r}}_{,\beta}\times(\check{\boldsymbol{\psi}}\times\mathbf{N}^\beta) = -(\check{\boldsymbol{\psi}}\times\bar{\mathbf{r}}_{,\beta})\times\mathbf{N}^\beta+\check{\boldsymbol{\psi}}\times(\bar{\mathbf{r}}_{,\beta}\times\mathbf{N}^\beta), \tag{47}$$

we finally obtain:

$$\begin{aligned}\mathrm{D}I[\mathfrak{u};\check{\mathfrak{u}},\hat{\mathfrak{u}}] = -\iint_M \{&[\check{\mathbf{N}}^\beta|_\beta+\check{\boldsymbol{\psi}}_{,\beta}\times\mathbf{N}^\beta+\check{\mathbf{p}}+\check{\boldsymbol{\psi}}\times\underline{(\mathbf{N}^\beta|_\beta+\mathbf{p})}]\cdot\hat{\mathbf{u}}\\ &+[\check{\mathbf{M}}^\beta|_\beta+\check{\boldsymbol{\psi}}_{,\beta}\times\mathbf{M}^\beta+\bar{\mathbf{r}}_{,\beta}\times\check{\mathbf{N}}^\beta+(\check{\mathbf{u}}_{,\beta}-\check{\boldsymbol{\psi}}\times\bar{\mathbf{r}}_{,\beta})\times\mathbf{N}^\beta+\check{\mathbf{l}}\\ &+\check{\boldsymbol{\psi}}\times\underline{(\mathbf{M}^\beta|_\beta+\bar{\mathbf{r}}_{,\beta}\times\mathbf{N}^\beta+\mathbf{l})}]\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}A\\ &+\int_{\partial M_f}\{[\check{\mathbf{N}}^\beta\nu_\beta-\check{\mathbf{N}}_\nu^*+\check{\boldsymbol{\psi}}\times\underline{(\mathbf{N}^\beta\nu_\beta-\mathbf{N}_\nu^*)}]\cdot\hat{\mathbf{u}}\\ &+[\check{\mathbf{M}}^\beta\nu_\beta-\check{\mathbf{M}}_\nu^*+\check{\boldsymbol{\psi}}\times\underline{(\mathbf{M}^\beta\nu_\beta-\mathbf{M}_\nu^*)}]\cdot\hat{\boldsymbol{\psi}}\}\,\mathrm{d}s. \end{aligned} \tag{48}$$

The underlined terms in (48) vanish identically whenever $\mathfrak{u}\in\mathrm{C}(C)$ is an equilibrium configuration. Consequently, the variational equation (26) implies the following buckling equations

$$\check{\mathbf{N}}^\beta|_\beta+\check{\mathbf{K}}_\beta\times\mathbf{N}^\beta+\check{\mathbf{p}} = 0, \tag{49a}$$

$$\check{\mathbf{M}}^\beta|_\beta+\check{\mathbf{K}}_\beta\times\mathbf{M}^\beta+\bar{\mathbf{r}}_{,\beta}\times\check{\mathbf{N}}^\beta+\check{\mathbf{E}}_\beta\times\mathbf{N}^\beta+\check{\mathbf{l}} = \mathbf{0}, \tag{49b}$$

with the corresponding static boundary conditions

$$\check{\mathbf{N}}^\beta\nu_\beta = \check{\mathbf{N}}_\nu^*,\quad \check{\mathbf{M}}^\beta\nu_\beta = \check{\mathbf{M}}_\nu^*\quad \text{along } \partial M_f. \tag{50}$$

In (49) we have introduced the kinematical relations (39). The above system of equations is to be supplemented by the homogeneous geometric boundary conditions

$$\check{\mathbf{u}} = \mathbf{0},\quad \check{\boldsymbol{\psi}} = \mathbf{0}\quad \text{along } \partial M_d. \tag{51}$$

5. SYMMETRY CONDITIONS

Of major importance in the analysis of buckling problems are the symmetry properties of the bilinear functional (46). To examine this problem we define the anti-symmetric part

$$\mathrm{D}I^a[\mathfrak{u};\check{\mathfrak{u}},\hat{\mathfrak{u}}] = \mathrm{D}I[\mathfrak{u};\check{\mathfrak{u}},\hat{\mathfrak{u}}]-\mathrm{D}I[\mathfrak{u};\hat{\mathfrak{u}},\check{\mathfrak{u}}], \tag{52}$$

of the bilinear functional (46) for kinematically admissible $\hat{\mathfrak{u}}, \check{\mathfrak{u}}\in\mathrm{TC}(C)$.

Let $\hat{\mathbf{E}}_\beta$ and $\hat{\mathbf{K}}_\beta$ denote the linearized strains associated with the displacement field $\hat{\mathbf{u}} \in TC(C)$. Using the constitutive relations (36) and some standard vector and tensor identities the anti-symmetric part (52) of the functional (46) can be expressed in the form

$$\mathrm{D}I^a[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}] = \iint_M \Sigma \,\mathrm{d}A + \iint_M \{\mathbf{M}^\beta \cdot (\hat{\boldsymbol{\psi}} \times \check{\boldsymbol{\psi}})|_\beta - (\bar{\mathbf{r}}_{,\beta} \times \mathbf{N}^\beta) \cdot (\hat{\boldsymbol{\psi}} \times \check{\boldsymbol{\psi}})\} \,\mathrm{d}A - \mathrm{D}I^a_{\mathrm{ext}}[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}], \tag{53}$$

where

$$\Sigma \equiv \mathbf{C}_1^{\alpha\beta} \cdot (\hat{\mathbf{E}}_\alpha \otimes \check{\mathbf{E}}_\beta - \check{\mathbf{E}}_\alpha \otimes \hat{\mathbf{E}}_\beta) + \mathbf{C}_2^{\alpha\beta} \cdot (\hat{\mathbf{E}}_\alpha \otimes \check{\mathbf{K}}_\beta - \check{\mathbf{E}}_\alpha \otimes \hat{\mathbf{K}}_\beta) + \mathbf{C}_3^{\alpha\beta} \cdot (\hat{\mathbf{K}}_\alpha \otimes \check{\mathbf{E}}_\beta - \check{\mathbf{K}}_\alpha \otimes \hat{\mathbf{E}}_\beta) + \mathbf{C}_4^{\alpha\beta} \cdot (\hat{\mathbf{K}}_\alpha \otimes \check{\mathbf{K}}_\beta - \check{\mathbf{K}}_\alpha \otimes \hat{\mathbf{K}}_\beta), \tag{54}$$

and

$$\mathrm{D}I^a_{\mathrm{ext}}[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}] = \iint_M \{\mathrm{D}\mathbf{p}[\mathbf{u};\check{\mathbf{u}}] \cdot \hat{\mathbf{u}} - \mathrm{D}\mathbf{p}[\mathbf{u};\hat{\mathbf{u}}] \cdot \check{\mathbf{u}} + \mathrm{D}\mathbf{l}[\mathbf{u};\check{\mathbf{u}}] \cdot \hat{\boldsymbol{\psi}} - \mathrm{D}\mathbf{l}[\mathbf{u};\hat{\mathbf{u}}] \cdot \check{\boldsymbol{\psi}}\} \,\mathrm{d}A + \int_{\partial M_f} \{\mathrm{D}\mathbf{N}^*_\nu[\mathbf{u};\check{\mathbf{u}}] \cdot \hat{\mathbf{u}} - \mathrm{D}\mathbf{N}^*_\nu[\mathbf{u};\hat{\mathbf{u}}] \cdot \check{\mathbf{u}} + \mathrm{D}\mathbf{M}^*_\nu[\mathbf{u};\check{\mathbf{u}}] \cdot \hat{\boldsymbol{\psi}} - \mathrm{D}\mathbf{M}^*_\nu[\mathbf{u};\hat{\mathbf{u}}] \cdot \check{\boldsymbol{\psi}}\} \,\mathrm{d}s \tag{55}$$

with $\mathrm{D}\mathbf{p}[\mathbf{u};\check{\mathbf{u}}]$, etc. defined by the formula (34). Applying next Stokes' theorem to the second surface integral in (53) and subsequently making use of the equilibrium equation (8b) we finally obtain

$$\mathrm{D}I^a[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}] = \iint_M \{\Sigma - \mathbf{l} \cdot (\hat{\boldsymbol{\psi}} \times \check{\boldsymbol{\psi}})\} \,\mathrm{d}A + \int_{\partial M_d} (\mathbf{M}^\beta \nu_\beta) \cdot (\hat{\boldsymbol{\psi}} \times \check{\boldsymbol{\psi}}) \,\mathrm{d}s + \int_{\partial M_f} \mathbf{M}^*_\nu \cdot (\hat{\boldsymbol{\psi}} \times \check{\boldsymbol{\psi}}) \,\mathrm{d}s - \mathrm{D}I^a_{\mathrm{ext}}[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}]. \tag{56}$$

Accordingly, at any equilibrium configuration $\mathbf{u} \in C(C)$,

$$\mathrm{D}I^a[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}] = 0 \text{ for any } \check{\mathbf{u}}, \hat{\mathbf{u}} \in TC(C) \text{ with } \check{\mathbf{u}} = \hat{\mathbf{u}} = \mathbf{o} \text{ on } \partial M_d \tag{57}$$

whenever the following conditions hold:

(a) $\Sigma = 0$ at any $\xi \in M$

(b) $\mathrm{D}I^a_{\mathrm{ext}}[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}] = 0$ for any $\check{\mathbf{u}}, \hat{\mathbf{u}} \in TC(C)$

(c) $\mathbf{l} = \mathbf{0}$ on M and $\mathbf{M}^*_\nu = 0$ along ∂M_f.

These conditions are sufficient for the bilinear functional (46) and (48), respectively, to be symmetric at any equilibrium configuration $\mathbf{u} \in C(C)$.

By virtue of (38), condition (a) is satisfied identically for hyperelastic shells. Conditions (b) and (c) indicate that the couple loadings acting on the shell are in general non-conservative. Moreover, our analysis clearly shows that at a non-equilibrium configuration the bilinear functional $\mathrm{D}I[\mathbf{u};\check{\mathbf{u}},\hat{\mathbf{u}}]$ is non-symmetric in general. The analogous result has been previously established by Simo and Vu-Quoc (1986) for spatial deformation of linearly elastic rods.

6. CLOSING REMARKS

The generality of the buckling equations derived in this paper enables the buckling analysis of shells undergoing an arbitrarily large elastic deformation. Some special problems have been considered in detail in Makowski and Stumpf (1989a, b). Here some additional remarks are pertinent.

As special cases the buckling equations for shells with a more restricted kinematical structure can be obtained. In particular, assuming that the rotation tensor $\mathbf{Q}$ is not an independent variable but is determined by the local deformation of the shell reference surface, the buckling equations can be reduced to those derived within shell theory based on the Kirchhoff–Love normality hypothesis. In this special case the transverse shear deformation is excluded but not the transverse normal deformation. Adopting Antman's terminology (Antman, 1976) we shall refer to this case as the unshearable shell model while the general theory of this paper will be called the shearable model.

The validity of the buckling equations presented in this paper is not limited by magnitude of strains, displacements and/or rotations, specific properties of a material or even the shell thickness. Their range of applicability will be solely confined by an accuracy of specific constitutive relations employed in the analysis.

A fairly general form of the constitutive equations for rubber-like shells is derived in Appendix B within a single kinematical assumption [cf. also Makowski and Stumpf (1986)]. These constitutive relations represent a substantial extension of earlier propositions in Biricikoglu and Kalnins (1971), Chernykh (1983) and Simmonds (1985) in a twofold sense. Firstly, they incorporate the transverse shear deformation. Secondly, they admit an arbitrary transverse normal deformation of the shell consistent with the incompressibility condition, while in the aforementioned papers power expansions with respect to the normal coordinate has been used retaining one, two or three terms. Moreover, in our derivation of the constitutive equations the reference surface may be arbitrarily allocated in the shell space without need of representing all relations in the form of power expansion with respect to the normal coordinate. This allows us to obtain the solutions for shells of virtually arbitrary thickness. For the problem of long circular cylindrical shells (circular rings) under external pressure, closed-form solutions of the governing buckling equations had been obtained in Makowski and Stumpf (1989a, b).

REFERENCES

Abraham, R. and Marsden, J. E. (1978). *Foundations of Mechanics*. 2nd Edn. Addison-Wesley, Reading, MA.

Antman, S. S. (1976). Ordinary differential equations of non-linear elasticity I: Foundations of the theories of non-linearly elastic rods and shells. *Arch. Rat. Mech. Anal.* **61**, 307–351.

Basar, Y. and Krätzig, W. B. (1985). *Mechanik der Flächentragwerke*. F. Vieweg und Sohn, Braunschweig.

Biricikoglu, V. and Kalnins, A. (1971). Large elastic deformations of shells with the inclusion of transverse normal strain. *Int. J. Solids Structures* **7**, 431–444.

Chernykh, K. F. (1983). The theory of thin shells of elastomers (in Russian). *Adv. Mech.* **6**, 111–147.

Cosserat, E. and F. (1909). *Théorie des Corps Déformables*. Herman et Fils, Paris.

Ericksen, J. L. and Truesdell, C. (1957). Exact theory of stress and strain in rods and shells. *Arch. Rat. Mech. Anal.* **1**, 295–323.

Green, A. E. and Naghdi, P. M. (1971). On superimposed small deformations on a large deformation of an elastic Cosserat surface. *J. Elasticity* **1**, 1–17.

Green, A. E., Naghdi, P. M. and Wainwright, W. L. (1965). A general theory of a Cosserat surface. *Arch. Rat. Mech. Anal.* **20**, 287–308.

Koiter, W. T. (1967). General equations of elastic stability for thin shells. *Proc. Symp. on the Theory of Shells*, University of Houston, Houston, Texas, pp. 187–228.

Libai, A. and Simmonds, J. G. (1983). Nonlinear elastic shell theory. *Advances in Applied Mechanics*, Vol. 23, pp. 271–371. Academic Press, New York.

Makowski, J. and Stumpf, H. (1986). Finite strains and rotations in shells. In *Finite Rotations in Structural Mechanics* (Edited by W. Pietraszkiewicz), Proc. EUROMECH Colloquium 197, pp. 175–194. Springer, Berlin.

Makowski, J. and Stumpf, H. (1989a). Finite axisymmetric deformation of shells of revolution with application to flexural buckling of circular plates. *Ing.-Archiv* **59**, 456–472.

Makowski, J. and Stumpf, H. (1989b). Buckling of nonlinearly elastic circular rings and arches (to be published).

Marsden, J. E. and Hughes, Th. J. R. (1986). *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliffs, New Jersey.

Naghdi, P. M. (1972). The theory of plates and shells. In *Handbuch der Physik* (Edited by S. Flügge). Vol. VIa/2, pp. 425–640. Springer, Berlin.

Pietraszkiewicz, W. (1984). Lagrangian description and incremental formulation in the non-linear theory of thin shells. *Int. J. Non-Linear Mech.* **19**, 115–140.
Reissner, E. (1974). Linear and nonlinear theory of shells. In *Thin-Shell Structures* (Edited by Y. C. Fung and E. E. Sechler), pp. 29–44. Prentice-Hall, Englewood Cliffs, New Jersey.
Simmonds, J. G. (1984). The nonlinear thermodynamical theory of shells: descent from 3-dimensions without thickness expansions. In *Flexible Shells* (Edited by E. L. Axelrad and F. A. Emmerling), pp. 1–11. Springer, Berlin.
Simmonds, J. G. (1985). The strain energy density of rubber-like shells. *Int. J. Solids Structures* **21**, 67–77.
Simo, J. C. and Vu-Quoc, L. (1986). Three-dimensional finite-strain rod model. Part II: Computational aspects. *Comp. Meth. Appl. Mech. Engng* **58**, 79–116.
Stumpf, H. (1981). On the linear and nonlinear stability analysis in the theory of thin shells. 15th IUTAM-Congress, Toronto 1980; *Ing. Arch.* **51**, 195–213.
Stumpf, H. (1984). On the non-linear buckling and post-buckling analysis of thin elastic shells. *Int. J. Non-Linear Mech.* **19**, 195–215.
Stumpf, H. and Makowski, J. (1986). On large strain deformation of shells. *Acta Mech.* **65**, 153–168.
Zubov, L. M. (1976). Theory of small deformations of prestressed thin shells (in Russian). *Prikl. Mat. Mekh.* **40**, 85–95.

APPENDIX A: DERIVATION OF THE EQUILIBRIUM EQUATIONS

The equilibrium equations for shells whose vectorial form is formally identical to that of (8) are well-known in the literature [cf. the discussion by Ericksen and Truesdell (1957) and Naghdi (1972)]. One has to note, however, that the methods of derivation vary throughout the literature and consequently the static variables entering these equations have different physical meaning. A brief derivation is included here not only for completeness but also for illustration that virtually contrasting arguments lead to the same form of the equilibrium equations. Moreover, the method of derivation employed here clearly shows that the equilibrium equations (8) are straightforward and exact implications of the three-dimensional balance laws.

I. *Direct approach* [*cf. Ericksen and Truesdell* (1957) *and Naghdi* (1972)]

In this approach a shell is regarded as the material surface S to which a microstructure may be ascribed. The basic postulates, not different in principle from their counterparts in continuum mechanics, are:

(a) external loadings acting on the shell consist of the surface force $\mathbf{p}$ and the surface couple $\mathbf{l}$ both defined per unit area of the undeformed configuration M of S.
(b) the action of the part of the shell outside any imagined, smooth, closed curve $\partial S'$ enclosing a subregion $S' \subset S$ on the part inside is equipollent to the stress resultant $\mathbf{N}_\nu$ and the couple resultant $\mathbf{M}_\nu$ defined per unit length of the image $\partial M'$ of $\partial S'$.

In consistence, the force $\mathbf{f}(S')$ and the torque $\mathbf{m}(S')$ acting on the part S' of the shell are given by

$$\mathbf{f}(S') = \int_{\partial M'} \mathbf{N}_\nu \,\mathrm{d}s + \iint_{M'} \mathbf{p}\,\mathrm{d}A,$$
$$\mathbf{m}(S') = \int_{\partial M'} (\mathbf{M}_\nu + \bar{\mathbf{r}} \times \mathbf{N}_\nu)\,\mathrm{d}s + \iint_{M'} (\mathbf{l} + \bar{\mathbf{r}} \times \mathbf{p})\,\mathrm{d}A. \tag{A1}$$

Here the external loadings and the stress and couple resultants are to be regarded as functions of the material coordinates $\boldsymbol{\xi} = (\xi^\beta)$. Moreover, as $\mathbf{N}_\nu$ and $\mathbf{M}_\nu$ are concerned one additional postulate is to be made (Cauchy's postulate in the continuum mechanics):

(c) if two curves $\partial M'$ and $\partial M''$ on M have a common normal direction at $\boldsymbol{\xi}$ in the tangent plane to M then

$$\mathbf{N}'_\nu(\boldsymbol{\xi}) = \mathbf{N}''_\nu(\boldsymbol{\xi}), \quad \mathbf{M}'_\nu(\boldsymbol{\xi}) = \mathbf{M}''_\nu(\boldsymbol{\xi}). \tag{A2}$$

Under suitable smoothness assumptions the postulate (A2) implies that

$$\mathbf{N}_\nu = \mathbf{N}_\nu(\boldsymbol{\xi}, \boldsymbol{\nu}(\boldsymbol{\xi})) = \mathbf{N}^\beta(\boldsymbol{\xi})\nu_\beta(\boldsymbol{\xi}),$$
$$\mathbf{M}_\nu = \mathbf{M}_\nu(\boldsymbol{\xi}, \boldsymbol{\nu}(\boldsymbol{\xi})) = \mathbf{M}^\beta(\boldsymbol{\xi})\nu_\beta(\boldsymbol{\xi}). \tag{A3}$$

Now introducing (A3) into (A1) and applying subsequently Stokes' theorem one finds

$$\mathbf{f}(S') = \iint_{M'} (\mathbf{N}^\beta|_\beta + \mathbf{p})\,\mathrm{d}A,$$
$$\mathbf{m}(S') = \iint_{M'} \{\mathbf{M}^\beta|_\beta + \bar{\mathbf{r}}_{,\beta} \times \mathbf{N}^\beta + \mathbf{l} + \bar{\mathbf{r}} \times (\mathbf{N}^\beta|_\beta + \mathbf{p})\}\,\mathrm{d}A. \tag{A4}$$

In view of the arbitrariness of $S' \subset S$ the static balance laws

$$\mathbf{f}(S') = \mathbf{0}, \quad \mathbf{m}(S') = \mathbf{0} \tag{A5}$$

imply the local equilibrium equations (8).

II. *Descent from three dimensions* [*cf. Libai and Simmonds* (1983), *Simmonds* (1984)]

Consider now a shell as a three-dimensional body $\mathscr{B}$ which, for simplicity, we may identify with its initial configuration $B \subset \mathscr{E}^3$. Let $\{\xi^i\} = \{\boldsymbol{\xi}, \zeta\}$ denote material coordinates taken to be normal ones in B and let us define the shell reference surface M by $\zeta = 0$. Assuming that $\zeta \in [-h_0^-, +h_0^+]$, the position vector of any particle in the initial configuration of the shell may be expressed in the form

$$\mathbf{X}(\boldsymbol{\xi}, \zeta) = \mathbf{r}(\boldsymbol{\xi}) + \zeta \mathbf{a}_3(\boldsymbol{\xi}). \tag{A6}$$

Under an arbitrary smooth deformation $\chi : B \to \mathscr{E}^3$ of the shell the same particle will move to a new place whose position vector may be represented by

$$\mathbf{x}(\boldsymbol{\xi}, \zeta) = \bar{\mathbf{r}}(\boldsymbol{\xi}) + \boldsymbol{\zeta}(\boldsymbol{\xi}, \zeta), \quad \boldsymbol{\zeta}(\boldsymbol{\xi}, 0) = 0 \tag{A7}$$

where $\bar{\mathbf{r}}$ denotes the position vector of the deformed reference surface $\bar{M} = \chi(M)$.

The force $\mathbf{F}(\mathscr{P})$ and the torque $\mathbf{M}(\mathscr{P})$ acting on any part $\mathscr{P} \subset \mathscr{B}$ of the shell are

$$\mathbf{F}(\mathscr{P}) = \int_{\partial P} \mathbf{Tn}\, \mathrm{d}S + \int_P \mathbf{f}\, \mathrm{d}V,$$
$$\mathbf{M}(\mathscr{P}) = \int_{\partial P} \mathbf{x} \times \mathbf{Tn}\, \mathrm{d}S + \int_P \mathbf{x} \times \mathbf{f}\, \mathrm{d}V, \tag{A8}$$

where $\mathbf{T}(\mathbf{X})$ denotes the first Piola–Kirchhoff stress tensor, $\mathbf{f}(\mathbf{X})$ is the body force and $\mathbf{n}(\mathbf{X})$ denotes the outward unit normal vector to the boundary ∂P of $P \subset B$. Assume now that P is obtained by normals to M along a smooth curve $\partial M'$ bounding $M' \subset M$. Then the boundary ∂P consists of three parts, the upper M^+ and lower M^- shell faces with the position vectors $\mathbf{X}^{\pm}(\boldsymbol{\xi}) = \mathbf{X}(\boldsymbol{\xi}, \pm h_0^{\pm})$ and the edge ∂P^* being the ruled surface with the position vectors $\mathbf{X}^*(s, \zeta) = \mathbf{r}(s) + \zeta \mathbf{a}_3(s)$. We have (Naghdi, 1972),

$$\mathbf{n}^{\pm}\, \mathrm{d}S^{\pm} = \pm(\mathbf{g}^3 \mp h^{\pm}_{0,\beta}\mathbf{g}^{\beta})^{\pm}\mu^{\pm}\, \mathrm{d}A,$$
$$\mathbf{n}^*\, \mathrm{d}S^* = \mathbf{g}^{\beta}\nu_{\beta}\mu\, \mathrm{d}\zeta\, \mathrm{d}s, \tag{A9}$$

where

$$\mathbf{g}_i = \mathbf{X}_{,i}, \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j,$$
$$\mu = 1 - 2\zeta H + \zeta^2 K, \quad \mathrm{d}V = \mu\, \mathrm{d}\zeta\, \mathrm{d}A. \tag{A10}$$

Using the expressions (A9) the force and the torque given by (A8) may be reduced to the form

$$\mathbf{F}(\mathscr{P}) = \int_{\partial M'} \mathbf{N}^{\beta}\nu_{\beta}\, \mathrm{d}s + \iint_{M'} \mathbf{p}\, \mathrm{d}A,$$
$$\mathbf{M}(\mathscr{P}) = \int_{\partial M'} (\mathbf{M}^{\beta} + \bar{\mathbf{r}} \times \mathbf{N}^{\beta})\nu_{\beta}\, \mathrm{d}s + \iint_{M'} (\mathbf{l} + \bar{\mathbf{r}} \times \mathbf{p})\, \mathrm{d}A, \tag{A11}$$

where the stress and couple resultants are defined by

$$\mathbf{N}^{\beta}(\boldsymbol{\xi}) = \int_-^+ \mathbf{T}^{\beta}\mu\, \mathrm{d}\zeta, \quad \mathbf{M}^{\beta}(\boldsymbol{\xi}) = \int_-^+ \boldsymbol{\zeta} \times \mathbf{T}^{\beta}\mu\, \mathrm{d}\zeta, \tag{A12}$$

and the statically equivalent surface loadings are

$$\mathbf{p} = \int_-^+ \mathbf{f}\mu\, \mathrm{d}\zeta + [(\mathbf{T}^3 \mp h^{\pm}_{0,\beta}\mathbf{T}^{\beta})\mu]^+_-,$$
$$\mathbf{l} = \int_-^+ \boldsymbol{\zeta} \times \mathbf{f}\mu\, \mathrm{d}\zeta + [\boldsymbol{\zeta} \times (\mathbf{T}^3 \mp h^{\pm}_{0,\beta}\mathbf{T}^{\beta})\mu]^+_-, \tag{A13}$$

where $\mathbf{T}^i = \mathbf{T}\mathbf{g}^i$ are the nominal stress vectors. Now the three-dimensional (static) balance laws $\mathbf{F}(\mathscr{P}) = \mathbf{0}$ and $\mathbf{M}(\mathscr{P}) = \mathbf{0}$ imply the local equilibrium equations (8) for shells.

APPENDIX B: DETERMINATION OF THE CONSTITUTIVE EQUATIONS

In Section 3 the constitutive equations had been assumed in the very general form (11). For rubber-like shells undergoing finite strain deformation we derived in Makowski and Stumpf (1986) constitutive equations by descent from the three-dimensional theory. Here a short review should expose and clarify some basic results of Stumpf and Makowski (1986) and Makowski and Stumpf (1986).

We regard the shell as a three-dimensional body whose particles are identified with their material coordinates

$\{\xi^i\} = \{\boldsymbol{\xi}, \xi\}$ taken to be normal ones in the initial configuration of the shell. Then the position vector of any point of the shell space may be expressed in the form

$$\mathbf{X}(\boldsymbol{\xi}, \xi) = \mathbf{r}(\boldsymbol{\xi}) + \xi \mathbf{a}_3(\boldsymbol{\xi}), \quad \xi \in [-h_0^-, +h_0^+] \tag{B1}$$

where $\mathbf{r}$ denotes the position vector of the undeformed reference surface M and $h_0 = h_0^+ + h_0^-$ is the initial shell thickness. The natural base vectors and the components of the metric tensor associated with (B1) are given by

$$\mathbf{g}_\alpha = \mu_\alpha^\beta \mathbf{a}_\beta, \quad \mathbf{g}^\alpha = (\mu^{-1})^\alpha_\beta \mathbf{a}^\beta, \quad \mathbf{g}_3 = \mathbf{g}^3 = \mathbf{a}_3, \tag{B2}$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \tag{B3}$$

where

$$\mu_\alpha^\beta = \delta_\alpha^\beta - \xi b_\alpha^\beta, \quad (\mu^{-1})^\alpha_\beta = \frac{1}{\mu}[\delta^\alpha_\beta - \xi(2H\delta^\alpha_\beta - b^\alpha_\beta)], \tag{B4}$$

$$\mu = \det \mu_\alpha^\beta = 1 - 2H\xi + K\xi^2, \tag{B5}$$

$$b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3, \quad H = \tfrac{1}{2} b^\beta_\beta, \quad K = \det b^\beta_\alpha. \tag{B6}$$

Various geometric quantities entering into these relations have standard meaning (Naghdi, 1972; Libai and Simmonds, 1983; Basar and Krätzig, 1985).

A two-dimensional strain energy function (per unit area of M) for the shell made of a hyperelastic material may formally be defined by

$$\Phi = \int_-^+ W(\mathbf{C})\mu \, d\xi, \quad \int_-^+ \equiv \int_{-h_0^-}^{+h_0^+}. \tag{B7}$$

Here $W(\mathbf{C})$ is a three-dimensional strain energy density, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ denotes the right Cauchy–Green deformation tensor and $\mathbf{F}$ is the deformation gradient. For a rubber-like material it is usually assumed that $W = W(I_1, I_2)$ and $I_3 = 1$ (incompressibility condition), where I_i, $i = 1, 2, 3$, are the principal invariants of $\mathbf{C}$. Denoting by $\mathbf{x}(\boldsymbol{\xi}, \xi)$ the position vector of a particle in the deformed configuration of the shell the incompressibility condition can be expressed in the form

$$\tfrac{1}{2}\varepsilon^{\alpha\beta}(\mathbf{x}_{,\alpha} \times \mathbf{x}_{,\beta}) \cdot \mathbf{x}_{,\xi} = \mu, \tag{B8}$$

where $(\ )_{,\xi}$ indicates partial differentiation with respect to the normal coordinate ξ. Moreover, the principal invariants I_1 and I_2 are given by

$$I_1 = g^{\alpha\beta}\bar{g}_{\alpha\beta} + \bar{g}_{33}, \quad I_2 = g_{\alpha\beta}\bar{g}^{\alpha\beta} + \bar{g}^{33}, \tag{B9}$$

where

$$\bar{g}_{\alpha\beta} = \mathbf{x}_{,\alpha} \cdot \mathbf{x}_{,\beta}, \quad \bar{g}_{\alpha 3} = \mathbf{x}_{,\alpha} \cdot \mathbf{x}_{,\xi}, \quad \bar{g}_{33} = \mathbf{x}_{,\xi} \cdot \mathbf{x}_{,\xi}, \tag{B10}$$

$$\bar{g}^{\alpha\beta} = \mu^{-2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\kappa}(\bar{g}_{\lambda\kappa}\bar{g}_{33} - \bar{g}_{\lambda 3}\bar{g}_{\kappa 3}), \tag{B11}$$

$$\bar{g}^{33} = \tfrac{1}{2}\mu^{-2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\kappa}\bar{g}_{\alpha\beta}\bar{g}_{\lambda\kappa}. \tag{B12}$$

From (12), (B7) and (B9)–(B12) it follows that the determination of the constitutive relations for rubber-like shells requires one to show that under suitable assumptions the components of the metric tensor (B10) can be expressed as functions of the strain measures $\mathbf{E}_\beta$, $\mathbf{K}_\beta$ and of the normal coordinate ξ such that the incompressibility condition (B8) is satisfied identically. The solution of this problem given below follows our earlier papers. Stumpf and Makowski (1986) and Makowski and Stumpf (1986).

Assume that the three-dimensional deformation of the shell is constrained such that the position vector $\mathbf{x}(\boldsymbol{\xi}, \xi)$ may be expressed in the form

$$\mathbf{x}(\boldsymbol{\xi}, \xi) = \bar{\mathbf{r}}(\boldsymbol{\xi}) + \zeta(\boldsymbol{\xi}, \xi)\mathbf{A}_3(\boldsymbol{\xi}), \quad \zeta(\boldsymbol{\xi}, 0) = 0. \tag{B13}$$

Here an unknown function ζ accommodates an arbitrary transverse normal deformation of the shell consistent with the kinematical constraint (B13). The differentiation of (B13) with the use of the definitions (6) of $\mathbf{E}_\beta$ and $\mathbf{K}_\beta$ yields

$$\begin{aligned}\mathbf{x}_{,\alpha} &= \mathbf{A}_\alpha + \mathbf{E}_\alpha + \zeta(\mathbf{K}_\alpha \times \mathbf{A}_3 - b^\beta_\alpha \mathbf{A}_\beta) + \zeta_{,\alpha}\mathbf{A}_3 \\ &= \{a_{\alpha\beta} + E_{\alpha\beta} - \zeta(b_{\alpha\beta} - K_{\alpha\beta})\}\mathbf{A}^\beta + (E_\alpha + \zeta_{,\alpha})\mathbf{A}_3, \\ \mathbf{x}_{,\xi} &= \zeta_{,\xi}\mathbf{A}_3,\end{aligned} \tag{B14}$$

where we have set

$$\mathbf{E}_\alpha = E_{\alpha\beta}\mathbf{A}^\beta + E_\alpha \mathbf{A}_3, \quad \mathbf{K}_\alpha = -\varepsilon^{\lambda\beta}K_{\alpha\beta}\mathbf{A}_\lambda + K_\alpha \mathbf{A}_3. \tag{B15}$$

Now introducing (B14) into (B8) the incompressibility condition reads

$$(1-2\mathscr{H}\zeta+\mathscr{K}\zeta^2)\frac{\partial\zeta}{\partial\xi}=\lambda_\xi(1-2H\xi+K\xi^2), \tag{B16}$$

where the following quantities have been introduced :

$$\begin{aligned}\lambda_\xi^{-1}(\boldsymbol{\xi}) &= \tfrac{1}{2}\varepsilon^{\alpha\beta}(\bar{\mathbf{r}}_{,\alpha}\times\bar{\mathbf{r}}_{,\beta})\cdot\mathbf{A}_3 = 1+a^{\alpha\beta}E_{\alpha\beta}+\tfrac{1}{2}\bar{\bar{E}}^{\alpha\beta}E_{\alpha\beta},\\ \mathscr{H}(\boldsymbol{\xi}) &= -\tfrac{1}{2}\lambda_\xi\varepsilon^{\alpha\beta}\bar{\mathbf{r}}_{,\alpha}\cdot(\mathbf{A}_{3,\beta}\times\mathbf{A}_3)\\ &= \lambda_\xi\{H-\tfrac{1}{2}a^{\alpha\beta}K_{\alpha\beta}+\tfrac{1}{2}\bar{\bar{E}}^{\alpha\beta}(b_{\alpha\beta}-K_{\alpha\beta})\},\\ \mathscr{K}(\boldsymbol{\xi}) &= \tfrac{1}{2}\lambda_\xi\varepsilon^{\alpha\beta}(\mathbf{A}_{3,\alpha}\times\mathbf{A}_{3,\beta})\cdot\mathbf{A}_3\\ &= \lambda_\xi\{K-\bar{\bar{K}}^{\alpha\beta}(b_{\alpha\beta}-\tfrac{1}{2}K_{\alpha\beta})\},\end{aligned} \tag{B17}$$

and

$$\bar{\bar{E}}^{\alpha\beta}=a^{\alpha\beta}a^{\lambda\kappa}E_{\lambda\kappa}-E^{\beta\alpha},\quad \bar{\bar{K}}^{\alpha\beta}=a^{\alpha\beta}a^{\lambda\kappa}K_{\lambda\kappa}-K^{\beta\alpha}. \tag{B18}$$

Recalling that $\zeta(\boldsymbol{\xi},0)=0$ the first-order differential equation (B16) can be integrated yielding

$$\mathscr{K}\zeta^3-3\mathscr{H}\zeta^2+3\zeta=\lambda_\xi\xi(3-3H\xi+K\xi^2). \tag{B19}$$

The general solution of this cubic algebraic equation is of the form

$$\zeta(\boldsymbol{\xi},\xi)=\zeta(\xi;\lambda_\xi,\mathscr{H},\mathscr{K};H,K). \tag{B20}$$

Now introducing (B20) into (B14) and (B9)–(B12) and subsequently into (B7) we obtain the expression for the strain energy Φ as a function of the strain measures $\mathbf{E}_\beta$ and $\mathbf{K}_\beta$. It is to be noted, however, that the presence of $\zeta_{,\alpha}$ in expression (B14) implies that Φ also depends upon the derivatives of $\mathbf{E}_\beta$ and $\mathbf{K}_\beta$ (Makowski and Stumpf, 1986). Therefore, in order to preserve the conventional structure of the shell theory considered in this paper we assume that $\zeta_{,\alpha}\cong 0$. This is not an essential restriction whenever the wavelength of the deformation pattern is sufficiently long. We next note that the strain energy function Φ derived within assumption (B13) does not depend on $K_\beta=\mathbf{K}_\beta\cdot\mathbf{A}_3$, i.e. the most general form of Φ is

$$\Phi=\Phi(E_{\alpha\beta},E_\alpha,K_{\alpha\beta};\boldsymbol{\xi}). \tag{B21}$$

Consequently, also the components $M^\beta=\mathbf{M}^\beta\cdot\mathbf{A}_3$ of the couples vanish, i.e. $M^\beta=0$.

The essential feature of our derivation of the strain energy function Φ and via Φ, the constitutive relations is this that a power expansion with respect to the normal coordinate ξ is not required. This fact may efficiently be used in the analysis of some special problems such as flexural buckling of circular plates or cylindrical deformation of infinite cylindrical shells [cf. Makowski and Stumpf (1989a, b)]. In the latter case the evaluation of the constitutive relations requires a numerical integration through the shell thickness. For relatively thin shells it is appropriate to represent the solution of the algebraic equation (B19) in the form

$$\begin{aligned}\zeta(\boldsymbol{\xi},\xi) &= \lambda_\xi(\xi+\tfrac{1}{2}\kappa_\xi\xi^2+\tfrac{1}{6}\chi_\xi\xi^3+\cdots),\\ \kappa_\xi &= 2(\lambda_\xi\mathscr{H}-H),\\ \chi_\xi &= 3\kappa_\xi(\kappa_\xi+2H)-2(\lambda_\xi^2\mathscr{K}-K),\ldots\end{aligned} \tag{B22}$$

Expressing also the remaining geometric quantities (B9)–(B12) in the form of power expansions with respect to ξ, different levels of approximations for Φ may be derived.

In the particular case when $\mathbf{A}_3$ is constrained to remain normal to the deformed reference surface the derived relations give the constitutive equations for the unshearable shell model. In this case $E_\alpha=0$ and $E_{\alpha\beta}=E_{\beta\alpha}$. Moreover, $\mathscr{H}$ and $\mathscr{K}$ have the meaning of the mean and Gaussian curvatures of the deformed reference surface of the shell.